

EFFECTIVE RESULTS FOR COMPLEX HYPERBOLIC MANIFOLDS

GABRIELE DI CERBO AND LUCA F. DI CERBO

ABSTRACT. The goal of this paper is to study the geometry of cusped complex hyperbolic manifolds through their compactifications. We derive effective very ampleness results for toroidal compactifications of finite volume complex hyperbolic manifolds. We give effective bounds on the number of complex hyperbolic manifolds with given upper bounds on the volume. Moreover, we estimate the number of ends of such manifolds in terms of their volume. Many of the statements in the case of surfaces are essentially sharp. The results of this work are based on the techniques developed in [DiC12].

CONTENTS

1. Introduction	1
1.1. Preliminaries	3
2. A gap theorem	4
3. Applications	7
3.1. Effective birationality	7
3.2. Bounds on the number of cusps	10
3.3. Bounds on the number of varieties	11
4. The two dimensional case	14
References	16

1. INTRODUCTION

Let \mathcal{H}^n be the complex n -dimensional hyperbolic space of dimension $n \geq 2$, that is, the Kähler space form with holomorphic sectional curvature -1 . Let X° be a complete non-compact complex hyperbolic manifold of finite volume. Such manifold is obtained as $X^\circ := \mathcal{H}^n \setminus \Gamma$ where Γ is torsion-free lattice of $\mathrm{PU}(1, n)$ with parabolic elements which is non-uniform. Then X° has finitely many disjoint unbounded ends of finite volume, the *cusps* of X° .

Baily-Borel [BB66] and Siu-Yau [SY82] proved that we can find a compactification X^* of X° such that the complement of X° in X^* consists of only finitely many (singular) points, called cusp points. Furthermore, under some mild assumptions on the lattice Γ , there exists a smooth variety X , which is a resolution of X^* , such that each exceptional divisor over a cusp point is a smooth abelian variety. For the precise requirements on Γ we refer the interested reader to Section 1.1. Let us denote by $D = \sum_i D_i$ the union of the exceptional divisors on X . We will refer

to the pair (X, D) as a toroidal compactification of X° . The dimension of X will always be denoted by n and assumed to be greater or equal than two. Such toroidal compactifications were first constructed by Mumford et al. [AMRT10] and by Mok [Mok12], for more details see again Section 1.1.

The goal of this work is to understand how the positivity properties of the log-canonical divisor $K_X + D$ determine the geometry of X° and (X, D) . The first technical result of this paper is that $K_X + D$ is not only big and nef but it is a limit of ample divisors of the form $K_X + \alpha D$ where α is a real number less than one. Moreover, there exists a uniform bound on the ampleness range for α depending on the dimension only. Remarkably, the existence of such a uniform bound is crucial for all the geometric applications presented in this work.

Theorem 1.1. *Let (X, D) be a toroidal compactification. Then $K_X + \alpha D$ is ample for all $\alpha \in (\frac{n-1}{n}, 1)$.*

The proof of Theorem 1.1 is based on the techniques developed by the authors in [DiC12]. We list here some of the main applications. We refer to Sections 3 and 4 for details.

Theorem 1.2. *Let (X, D) be a toroidal compactification. Then for all $m \geq (n+1)^4$, the map associated to $|m(K_X + D)|$ is a morphism and it defines an embedding of $X \setminus D$ into some projective space. Furthermore, this morphism maps all the different components of D to distinct singular points.*

Theorem 1.2 is an effective version of a recent result of Mok, see Main Theorem in [Mok12]. Nevertheless, our proof relies on completely different techniques than Mok's. Note that when (X, D) is a toroidal compactification associated to a neat arithmetic lattice in $\mathrm{PU}(1, n)$, Theorem 1.2 gives a very concrete realization of the classical Baily-Borel compactification X^* as a projective algebraic variety.

Theorem 1.2 can also be applied to study the geometry of the smooth variety X in the pair (X, D) .

Theorem 1.3. *Let (X, D) be a toroidal compactification. Then X can be realized as a smooth subvariety of \mathbb{P}^{2n+1} such that its degree d satisfies*

$$d \leq (n+1)^{4n} (K_X + D)^n.$$

A similar result is obtained by Hwang in [Hwa05]. The main advantage of our approach is that it provides bounds which are linear in $(K_X + D)^n$. The bounds obtained by Hwang are polynomial of degree $n+1$ in $(K_X + D)^n$.

Theorem 1.3 has several applications. For example, combining this result with classical Chow variety techniques, we can derive explicit bounds for the number of complex hyperbolic manifolds with given upper bounds on the volume. These results are effective versions of the classical Wang's finiteness theorem, see [Wan72]. For the numerical values of these bounds see Corollaries 3.14 and 3.17.

As a final application of Theorem 1.1, we show how to bound the number of cusps of a finite volume complex hyperbolic manifold in terms of its volume.

Theorem 1.4. *Let (X, D) be a toroidal compactification. Let q be the number of cusps of X° . Then*

$$q \leq n^n (K_X + D)^n.$$

The problem of bounding the number of ends in pinched negatively curved finite volume manifolds is a long standing problem in differential geometry. For complex hyperbolic manifolds, the classical approach is via techniques coming from geometric topology, see [Par98] and the bibliography therein. More recently, Hwang in [Hwa04] was able to improve Parker's result by using the Hirzebruch-Mumford proportionality principle [Mum77]. The bound we present here in Theorem 1.4 is better than the one obtained by Parker but worse than Hwang's for $n \geq 3$. We decided to include this result here mainly for two reasons. First, the proof of Theorem 1.4 is completely elementary and it follows easily from Theorem 1.1. Second, the reasoning given in the proof of Theorem 1.4 is in principle applicable in a much general setting while both the approaches of Parker and Hwang use in an essential way the special properties of complex hyperbolic manifolds.

In Section 3.2, we describe an alternative approach to the problem of bounding the cusps, see Proposition 3.12. This bound is asymptotically better than the one given in Theorem 1.4 but still weaker than Hwang's for $n \geq 4$. Nevertheless, it appears to be the best currently known bound for threefolds.

Finally, in Section 4 we derive much sharper versions of Theorems 1.2, 1.3 and 1.4 in the case of complex surfaces. These improvements rely on the fact that we are able to derive an essentially sharp version of Theorem 1.1 for toroidal compactification of dimension two, see Theorem 4.2 in Section 4. Moreover, we are able to further improve the bound on the number of cusps given in Theorem 1.4. This result seems to be the best bound for surfaces currently available in the literature.

1.1. Preliminaries. The theory of compactifications of locally symmetric varieties has been extensively studied, see for example [BJ06]. For technical reasons this theory is mainly developed for quotients of symmetric spaces by arithmetic subgroups. In most cases this is not a serious issue since the work of Margulis [Mar84] implies that lattices in any semi-simple Lie group of real rank bigger or equal than two are arithmetic subgroups. Nevertheless, this theorem does not cover the case of lattices in the complex hyperbolic space \mathcal{H}^n which are the main object of study in this paper. Note that non-arithmetic lattices in $\mathrm{PU}(1, n)$ were constructed by Mostow and Deligne-Mostow; see [DM93] and the extensive bibliography there.

It would then be desirable to develop a theory of compactifications of finite volume complex hyperbolic manifolds which does not rely on the arithmeticity of the defining torsion free lattices. Fortunately, this problem was addressed by many mathematicians from several different point of views. A compactification of finite volume complex hyperbolic manifolds as a complex spaces with isolated normal singularities was obtained by Siu and Yau in [SY82]. This compactification may be regarded as a generalization of the Baily-Borel compactification defined for arithmetic lattices in \mathcal{H}^n . A toroidal compactification for finite volume complex hyperbolic manifolds was described by Hummel and Schroeder in [HS96]. In many cases these

compactifications provide explicit resolution of singularities of the Siu-Yau compactifications. More recently Mok [Mok12] gave a lucid and detailed description of these compactifications and described many of their remarkable features.

The compactifications studied in this work are the ones described by Hummel-Schroeder and Mok. Thus, we do not require that lattice Γ to be arithmetic. On the other hand, we require that all the parabolic isometries of Γ are *unipotent*, in other words we require that they act by translations on their invariant horospheres. We impose this condition in order to obtain smooth toroidal compactifications. We would like to point out that this technicality is hidden in the construction described by Mok while it is explicitly discussed in the work of Hummel-Schroeder. Let us describe in more details this technical point. Recall that given a non-uniform torsion-free lattice $\Gamma \leq \mathrm{PU}(1, n)$, the finite volume complex hyperbolic manifold \mathcal{H}^n/Γ has finitely many cusps C_1, \dots, C_m which are in one to one correspondence with the maximal parabolic subgroups of Γ , see [Ebe96] for more details. Given a cusp C_i , denote by $\Gamma_i \leq \Gamma$ the associated maximal parabolic subgroup and by HB_i the horoball stabilized by Γ_i . After choosing an Iwasawa decomposition [Ebe96] for $\mathrm{PU}(1, n)$, we can identify $\partial\mathrm{HB}$ with a Heisenberg type Lie group N_i diffeomorphic to $\mathbb{C}^{n-1} \times \mathbb{R}$. Thus, the center Z_i of N_i is $Z_i = [N_i, N_i]$ and it is isomorphic to \mathbb{R} . Furthermore, the simply connected Lie group N_i comes equipped with a natural left invariant metric and then we can consider Γ_i as a lattice in $\mathrm{Iso}(N_i)$. The isometry group of N is isomorphic to the semi-direct product $\mathrm{Iso}(N_i) = N_i \rtimes U(n-1)$. We then say that $\phi \in \Gamma_i$ is unipotent if it is a translation in $\mathrm{Iso}(N_i)$. Now, the construction described by Hummel-Schroeder and Mok produces a smooth compactification if the quotients $\Gamma_i \backslash \Gamma_i \cap Z_i$ are torsion free for all i . This is the case if all the parabolic isometries of Γ are unipotent. In the classical arithmetic case this requirement is usually satisfied by requiring the lattice Γ to be neat, see [BB66] and [BJ06] for more details. Let us note that given any non-uniform lattice $\Gamma \in \mathrm{PU}(1, n)$, there always exists a finite index subgroup whose all parabolic isometries are unipotent, in the arithmetic case see [AMRT10] while for the general non-arithmetic case we refer to [Hum98].

Acknowledgements. The first named author would like to express his gratitude to Professor János Kollár for his constant support and for valuable discussions.

2. A GAP THEOREM

In this section we prove Theorem 1.1 which is the main technical result of the paper. The proof of Theorem 1.1 is based on the following result contained in [DiC12], see Theorem 4.15.

Theorem 2.1. *Let X be a smooth projective variety and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Then $K_X + \alpha D$ is ample for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

The key ingredient in the proof of Theorem 2.1 is the Cone Theorem. In particular the fact that $\alpha \in (\frac{n+1}{n+2}, 1)$ is obtained using the bound on the length of extremal rays given in the Cone Theorem. Let us recall the definition of an extremal ray. We refer to [Kol96] and [KM98] for more details.

Definition 2.2. *Let $N \subset \mathbb{R}^m$ be a cone. A subcone $M \subset N$ is called extremal if $u, v \in N$, $u + v \in M$ imply that $u, v \in M$. A 1-dimensional extremal subcone is called an extremal ray.*

In this paper we will consider only extremal rays R of the cone of effective 1-cycles $\overline{NE}(X)$ such that $K_X \cdot Z < 0$ for any effective 1-cycle Z in R . For such extremal rays we define the length of R as

$$l(R) := \min \{ -K_X \cdot C \mid C \text{ is a rational curve with numerical class in } R \}.$$

The length of extremal rays in $\overline{NE}(X)$ has been extensively studied. The Cone Theorem gives that $l(R) \leq n + 1$. On the other hand, there are many classification results regarding varieties with extremal rays of maximal length. The result we need here is due to Wiśniewski [Wis89]. Even though it is not the most general result currently known, it is enough for our purposes.

Theorem 2.3 (Wiśniewski). *Let X be smooth projective variety of dimension n .*

- (1) *If there exists an extremal ray R of length $n + 1$, then $\text{Pic}(X) = \mathbb{Z}$ and $-K_X$ is an ample line bundle on X .*
- (2) *If there exists an extremal ray R of length n then, either $\text{Pic}(X) = \mathbb{Z}$ and $-K_X$ is ample or $\rho(X) = 2$ and there exists a morphism $\text{contr}_R : X \rightarrow B$ onto a smooth curve B whose general fiber is a smooth $(n - 1)$ -manifold that satisfies the conditions in (1).*

Our goal is to prove that toroidal compactifications do not fit in the above list of varieties. We start with a lemma which rules out most of the cases.

Lemma 2.4. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then $\rho(X) > q$, where $\rho(X)$ is the Picard number of X .*

Proof. Suppose by contradiction that $\rho(X) \leq q$. Let H be an ample divisor on X . Then there exists a non trivial relation

$$a_0 H + \sum_{i=1}^q a_i D_i \equiv 0,$$

where $a_i \in \mathbb{R}$. Recall that $D_i^n < 0$ and $D_i \cdot D_j^{n-1} = 0$ for all $j \neq i$. Intersecting the above relation with $a_i D_i^{n-1}$ we get that $a_0 a_i > 0$ for all i . On the other hand

$$0 = a_0 H^{n-1} \cdot \left(a_0 H + \sum_{i=1}^q a_i D_i \right) = a_0^2 H^n + \sum_{i=1}^q a_0 a_i D_i \cdot H^{n-1} > 0,$$

gives a contradiction. \square

In particular, if a toroidal compactification has a least two cusps, then $l(R) \leq n - 1$ for all extremal rays. It is left to prove that a toroidal compactification with

only one cusp does not fit in the cases described by Theorem 2.3. This is the content of the next lemma.

Lemma 2.5. *Let (X, D) be a toroidal compactification with $\dim(X) = n$. Then any extremal ray R of X has length at most $n - 1$.*

Proof. We want to rule out the cases in Theorem 2.3. By Lemma 2.4 we can assume that D has only one component and $\rho(X) = 2$. In particular X does not have any extremal ray of length $n + 1$. Suppose by contradiction that we have an extremal ray of length n . Thus, Theorem 2.3 implies that there exists a fibration $\pi : X \rightarrow B$ over a smooth curve B .

We first deal with the case $\dim(X) = 2$. Since D is an elliptic curve, we have that π restricted to D is a surjective morphism and in particular $g(B) \leq 1$. Since $\rho(X) = 2$, the ruled surface X must be minimal. If B is a rational curve then X is a Hirzebruch surface. On a Hirzebruch surface the only curve with negative self intersection is the zero section, which is a rational curve, so it cannot be D .

We can assume that B is an elliptic curve. We follow here the notation in [Har77]. Write $X = \mathbb{P}(\mathcal{E})$ where \mathcal{E} is a normalized rank 2 vector bundle over B with invariant e . By a theorem of Atiyah we know that $e \geq -1$, see Theorem V.2.15 in [Har77]. If $e \leq 0$ then $-K_X$ is nef and in particular by adjunction $D^2 \geq 0$. We then have $e \geq 1$. A simple computation using Proposition V.2.20 in [Har77] shows that the only curve with negative self-intersection is the zero section. This implies that D is the zero section. We can then find a \mathbb{P}^1 with just one puncture in $X \setminus D$. Since curvature can only decrease along complex submanifolds of a negatively curved Kähler manifold, we obtain a contradiction.

Now suppose that $\dim(X) \geq 3$. By a dimension argument, there exists a curve C in $D \cap F$, where F is the general fiber of π . We compute $(K_X + D) \cdot C$ in two different ways to get a contradiction. First, since $C \subset D$, we know that $(K_X + D) \cdot C = 0$. On the other hand, since the normal bundle of D in X is anti-ample, we have $D \cdot C < 0$. Moreover, since F is a Fano variety $K_X \cdot C = K_F \cdot C < 0$. We then conclude $(K_X + D) \cdot C < 0$. This is a contradiction. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1. First, we would like to show that $K_X + D$ is big and nef. Recall that \mathcal{H}^n is the negatively curved complex space form. Thus, any manifold $X^\circ = \mathcal{H}^n \setminus \Gamma$, with Γ non-uniform, is equipped with a standard negatively curved Kähler-Einstein metric of finite volume which we denote by $\hat{\omega}$. By Théorème 1.1 in [Sib85], such a metric $\hat{\omega}$ on X° can be regarded as a closed *positive* current on X . Moreover, it is not difficult to see that $\hat{\omega}$ is in the cohomology class of $K_X + D$. More precisely, let D_i be the irreducible components of D and let $s_i \in \mathcal{O}_X(D_i)$ be the defining sections. By appropriately choosing Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}_X(D_i)$, the volume form associated to $\hat{\omega}$ is then of the form

$$\Psi = \frac{\Omega}{\prod_i \|s_i\|^2 (-\log \|s_i\|^2)^{n+1}}$$

for some globally defined volume form Ω on X . Thus, the Poincaré-Lelong formula combined with the fact that $\hat{\omega}$ is Kähler-Einstein gives that $\hat{\omega} \in [K_X + D]$. Next, let us observe that $\omega_P \geq \hat{\omega}$ for some Kähler current ω_P on X with Poincaré type singularities along D . We then have that $\hat{\omega}$ is a closed positive current with zero Lelong numbers on X . Moreover, since $\hat{\omega}$ is a regular Kähler metric on X° , it is strictly positive at any point $p \in X \setminus D$. By using a regularization argument based on results of Demailly [Dem92], we conclude that $K_X + D$ is big and nef, see also Theorem 1.3 in [JS93]. We can actually conclude more. In fact, we must have $(K_X + D)^{\dim(Z)} \cdot Z > 0$ for any subvariety Z not contained in D .

Finally, we want to show that if C is a curve such that $(K_X + D) \cdot C = 0$ then $K_X \cdot C > 0$. By the discussion above we know that C must be contained in D . Then

$$K_X \cdot C = -D \cdot C > 0,$$

since the normal bundle of D in X is negative, see Theorem 1 in [Mok12]. In particular Theorem 2.1 implies that $K_X + \alpha D$ is ample for all values of α close enough to one. Following the proof of Theorem 2.1 in [DiC12] and using the bound given by Lemma 2.5, it follows that the $K_X + \alpha D$ is ample for all $\alpha \in (\frac{n-1}{n}, 1)$. \square

It seems a difficult problem to understand which varieties arise as toroidal compactifications of hyperbolic manifolds. The above theorem gives a first step toward a possible solution of the problem.

Corollary 2.6. *There are no toroidal compactification (X, D) with X a smooth Fano variety.*

Proof. Suppose (X, D) is a toroidal compactification with $-K_X$ ample. Because of Theorem 1.1, for all α close to one, we know $K_X + \alpha D$ is ample. By Corollary 4.18 in [DiC12], it follows that $K_X + D$ is strictly nef. On the other hand, we must have $(K_X + D)|_D = \mathcal{O}_D$. This is a contradiction. \square

Question 2.7. *It is interesting to ask whether there exists a smooth toroidal compactification of a ball quotient with negative Kodaira dimension.*

3. APPLICATIONS

In this section, we give the proofs of Theorems 1.2, 1.3 and 1.4 stated in the Introduction.

3.1. Effective birationality. Let us start by studying the birational properties of the divisor $K_X + D$. First, we prove that the map associated to $|m(K_X + D)|$ maps the components of D to distinct points for any $m \geq 2$.

Proposition 3.1. *Let (X, D) be a toroidal compactification. Then for any i there exists a section σ_i of $H^0(X, \mathcal{O}_X(2(K_X + D)))$ such that $\sigma_i|_{D_i} \neq 0$ and $\sigma_i|_{D_j} = 0$ for all $j \neq i$.*

Proof. Write $D = \sum_{i=1}^q D_i$ and recall that each component is an abelian variety and they are all disjoint. Consider the following exact sequence

$$0 \rightarrow \mathcal{O}_X(2K_X + D) \xrightarrow{D} \mathcal{O}_X(2(K_X + D)) \rightarrow \mathcal{O}_D \rightarrow 0.$$

By Kawamata-Viehweg's vanishing we have that $H^1(X, \mathcal{O}_X(2K_X + D)) = 0$. Thus, taking the long exact sequence in cohomology, we get the following surjective map

$$H^0(X, \mathcal{O}_X(2(K_X + D))) \rightarrow \bigoplus_{i=1}^q H^0(D_i, \mathcal{O}_{D_i}).$$

□

The next step is to understand what happens outside the boundary divisor. Let us start by deriving a lower bound on the top self-intersection of $K_X + D$.

Lemma 3.2. *Let (X, D) be a toroidal compactification. Then*

$$(K_X + D)^n \geq (n+1)^{n-1}.$$

Proof. Since the Kähler-Einstein current $\hat{\omega}$ can be multiplied by itself, the top self-intersection of $L := K_X + D$ can be expressed in terms of the Riemannian volume of X^o . More precisely, by normalizing the holomorphic sectional curvature to be -1 we have

$$\text{Vol}(X) = \frac{(4\pi)^n}{n!(n+1)^n} L^n.$$

On the other hand, Gromov-Harder's generalization of Gauss-Bonnet [Gro82] implies that

$$\text{Vol}(X) \geq \frac{(4\pi)^n}{(n+1)!}.$$

Combining the two formulas above we get the result.

□

These considerations and a theorem of Kollár in [Kol97] imply the following.

Corollary 3.3. *Let (X, D) be a toroidal compactification. Then $m(K_X + D)$ is base point free for any $m \geq \binom{n}{2} + 1$.*

Proof. By Theorem 1.1, for any subvariety Z not contained in D we have $(K_X + D)^{\dim(Z)} \cdot Z > 0$. Because of Theorem 5.8 in [Kol97], we know that $m(K_X + D)$ is free at all points outside D for any $m \geq \binom{n}{2} + 1$. Moreover, by Lemma 3.1 we know already that $2(K_X + D)$ is free on the divisor D .

□

Similarly, we can study separation of points.

Corollary 3.4. *Let (X, D) be a toroidal compactification. Then $m(K_X + D)$ separates any two points in $X \setminus D$ for any $m \geq \binom{n}{2} + 2$.*

Proof. Combine Theorem 5.9 in [Kol97] and Lemma 3.2.

□

Since $X \setminus D$ is open in X , Corollary 3.4 implies that $|m(K_X + D)|$ defines a birational map for any $m \geq \binom{n}{2} + 2$. These simple results can be already used to slightly improve all the bounds in [Hwa05]. Nevertheless, the approach we follow here is quite different. Note that Corollary 3.4 deals only with separation of points. Thus, if we want to prove that $m(K_X + D)$ defines an embedding outside D we need something more. Again the key for us is Theorem 1.1, while the approach described in [Hwa05] relies on Seshadri constants type arguments. Let us start by recalling the following definition.

Definition 3.5. *Let X be a smooth projective variety and let D be an effective divisor. We say that a divisor L is very ample modulo D if the map $\phi_L : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(L)))$ defines an embedding of $X \setminus D$.*

We can now derive an effective result on very ampleness modulo D of $K_X + D$. In order to keep the final statement simple, some of the constants used in the proof are not optimal.

Theorem 3.6. *Let (X, D) be a toroidal compactification of dimension n . Then $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 1)^4$.*

Proof. Theorem 1.1 implies that $K_X + \frac{n}{n+1}D$ is ample. Then $H := (n+1)K_X + nD$ is an ample integral divisor. By a theorem of Angehrn-Siu we have that $K_X + mH$ is ample and base point free for any $m > \binom{n+1}{2}$, see [AS95] or [Kol97] and [Laz04b] for an algebraic proof. In particular $B := K_X + (n^2 + 1)H$ is ample and base point free. A corollary of Castelnuovo-Mumford regularity, see Example 1.8.23 in [Laz04a], gives that $K_X + (n+2)B$ is very ample. All together we have that the divisor

$$((n^2 + 1)(n + 1)(n + 2) + n + 3) K_X + (n^2 + 1)n(n + 2)D$$

is very ample. Let $M := ((n^2 + 1)(n + 1)(n + 2) + n + 3)$. Adding the right positive multiple of D we get the following injective map

$$H^0(X, \mathcal{O}_X(K_X + (n + 2)B)) \hookrightarrow H^0(X, \mathcal{O}_X(M(K_X + D))).$$

This implies that $m(K_X + D)$ is very ample modulo D for any $m \geq M$. Since $(n + 1)^4 \geq M$ we get the statement of the theorem. \square

Theorem 1.2 stated in the introduction is now just a combination of Theorem 3.6 and Proposition 3.1.

Let us conclude this section by discussing the connections between Fujita's conjecture and Theorem 3.6. First, recall that if $\dim(X) \leq 4$ then the base point free part of Fujita's conjecture is known to be true thanks to the work of Kawamata [Kaw97]. We can then improve the bound in Theorem 3.6 for low dimensional varieties.

Corollary 3.7. *Let (X, D) be a toroidal compactification of dimension $n \leq 4$. Then $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 2)^3$.*

Proof. It follows along the same lines of Theorem 3.6. Instead of using Angehrn-Siu's theorem, we use Kawamata's result on Fujita's conjecture. The rest remains unchanged. \square

It is interesting to note that using a theorem of G. Heier in [Hei02], we can get a bound of order $n^{10/3}$ in Theorem 3.6. Of course an even better bound would be obtained using the very ampleness part of Fujita's conjecture. More precisely, assuming the conjecture of Fujita to be true, the proof of Theorem 3.6 easily gives that $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 2)^2$.

3.2. Bounds on the number of cusps. In this section, we show how to explicitly estimate the number of cusps of X° in terms of the intersection number $(K_X + D)^n$. We present two different approaches. The first one gives asymptotically a worst bound. Nevertheless, it can be in principle applied to a much more general setting and, with suitable improvements, it gives the best known bound in the case of surfaces, see Section 4 for details. The second method gives a better bound for $n \geq 3$ and it follows easily from a result of Matsusaka. Moreover, it gives the best currently known bound for threefolds.

Recall that X° has finitely many ends which correspond to the cusp points of X^* . Each cusp point gives rise to one component of the boundary divisor D in X . So bounding the number of cusps is equivalent to bounding the number of components of D .

We can now prove Theorem 1.4 stated in the Introduction.

Theorem 3.8 (Theorem 1.4). *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq n^n (K_X + D)^n.$$

Proof. Once we have Theorem 1.1, the above result follows from a quite straightforward computation. Let $L := K_X + D$. Because of Theorem 1.1, we know that $nL - D$ is nef and $(n + 1)L - D$ is ample. Then

$$q \leq D \cdot ((n + 1)L - D)^{n-1} = (-1)^{n-1} D^n.$$

Since $nL - D$ is nef we get that

$$(nL - D)^n = n^n L^n + (-1)^n D^n \geq 0.$$

Combining the two inequalities we get the result. \square

Note that we can also a bound on the top self-intersection of D in terms of $(K_X + D)^n$ only. Let us summarize this fact in the form of a corollary as it will be used in Section 3.3.

Corollary 3.9. *Let (X, D) be a toroidal compactification. Then*

$$(-1)^{n-1} D^n \leq n^n (K_X + D)^n.$$

We now present an estimate on the number of cusps which does not rely on Theorem 1.1. The key point is the following corollary of Proposition 3.1.

Corollary 3.10. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$h^0(X, \mathcal{O}_X(2(K_X + D))) \geq q.$$

In particular, in order to bound the number of cusps, it is enough to bound $h^0(X, \mathcal{O}_X(2(K_X + D)))$. This can be done using a result of Matsusaka. For an outline of the proof we refer to [Kol96].

Theorem 3.11 (Matsusaka). *Let H be a big and nef divisor on a smooth variety X of dimension n . Then for any $m \geq 1$ we have*

$$h^0(X, \mathcal{O}_X(mH)) \leq m^n H^n + n.$$

Combining these two results we get the following.

Proposition 3.12. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq (2^n + 1)(K_X + D)^n.$$

Proof. By Lemma 3.2, we have that $(K_X + D)^n \geq n$. □

It is interesting to note how the bounds given in Theorem 1.4 and Proposition 3.12 rely on few basic geometric properties of the pair (X, D) . Roughly speaking, they depend on the fact that $L = K_X + D$ is big, nef and such that $L|_D = \mathcal{O}_D$. None of the other special features of toroidal compactifications of ball quotients is used. Therefore, these techniques can be successfully applied to study complete finite volume Kähler manifolds with pinched negative sectional curvature which are not locally symmetric. These results will appear elsewhere.

Finally, despite their simple nature, the bounds given in Theorem 1.4 and Proposition 3.12 are respectively the best currently known bounds for surfaces and three-folds, compare with [Par98] and [Hwa04]. In the case of surfaces, see also the improvement given in Proposition 4.3 below.

3.3. Bounds on the number of varieties. In this section we study the problem of finding effective bounds on the number of complex hyperbolic manifolds with bounded volume. The main issue is to find effective embedding results for toroidal compactifications. This was successfully done in Theorem 3.6 and we investigate its consequences here. The key result of this section is Theorem 1.3 stated in the Introduction, which we restate for the convenience of the reader.

Theorem 3.13 (Theorem 1.3). *Let (X, D) be a toroidal compactification. Then X can be realized as a smooth subvariety of \mathbb{P}^{2n+1} such that its degree d satisfies*

$$d \leq (n + 1)^{4n} (K_X + D)^n.$$

Proof. Let $L := K_X + D$. By Theorem 3.6, we know that $(n + 1)^4 L - kD$ is very ample, for some explicit positive integer k . Then $(n + 1)^4 L - kD$ defines an embedding of X into some projective space with degree $d \leq (n + 1)^{4n} L^n$. By general projection, we obtain that X sits inside \mathbb{P}^{2n+1} . □

As pointed out in the Introduction, Hwang proved a similar result in [Hwa05]. While our bound is linear in $(K_X + D)^n$, he derived a polynomial bound in $(K_X + D)^n$ of degree $n + 1$. Once we have Theorem 3.13, counting the number of complex hyperbolic manifolds is reduced to standard techniques on the complexity of Chow varieties. Let us briefly explain this point. Associated to a hyperbolic manifold, we have a toroidal compactification (X, D) . Therefore, it is enough to count such pairs. Fujiki in [Fuj92] proved that a toroidal compactification (X, D) is infinitesimally rigid under deformations of the pair. This implies that the number of toroidal compactifications is bounded by the number of components of a suitable Chow variety. See Corollary 3.17. Moreover, Hwang pointed out that Fujiki's theorem implies something more. If $n \geq 3$, X itself is rigid under deformations and the same method applies, see Proposition 4.2 in [Hwa05].

We start by counting the varieties which arise as toroidal compactifications of complex hyperbolic manifolds with bounded volume. In particular, in the next statement we forget about the extra structure coming from the boundary divisor.

Fix two positive integers d and $m > n$. We denote by $\text{Chow}_m(n, d)$ the Chow variety of n -dimensional irreducible smooth subvarieties of \mathbb{P}^m of degree d .

Corollary 3.14. *Fix two positive integers $n \geq 3$ and V . Then the number of varieties X arising as toroidal compactifications (X, D) with $\dim(X) = n$ and $(K_X + D)^n \leq V$ is bounded by*

$$\sum_{d=1}^{d_0} \binom{(2n+2)d}{2n+1}^{(2n+2)d \binom{d+n-1}{n} + (2n+2) \binom{d+n-1}{n-1}},$$

where $d_0 = (n+1)^{4n} V$.

Proof. Let (X, D) be a toroidal compactification as in the statement. By Theorem 3.13, X can be embedded in \mathbb{P}^{2n+1} as a smooth subvarieties of degree $d \leq d_0$. Proposition 4.2 in [Hwa05] implies that X is rigid under deformation. In particular, the number of such toroidal compactifications is bounded by the number of components of the Chow variety $\text{Chow}_{2n+1}(n, d_0)$. A straightforward application of I.3.28.9 in [Kol96] gives the result. \square

Remark 3.15. *Independently of the value of d_0 , the bound on the number of components of the Chow variety is of the form*

$$(a_n d_0)^{(b_n d_0)^{n+1}},$$

for some function a_n and b_n which depend only on n . Therefore, the bound in Corollary 3.14, expressed just in terms of the volume, is of the form

$$V^{V^{n+1}}.$$

We can now give effective estimates on the number of toroidal compactifications (X, D) with bounded $(K_X + D)^n$, counted as pairs. This corresponds exactly to bounding the number of complex hyperbolic manifolds with bounded volume. Unfortunately the bounds are slightly worst than the ones in Corollary 3.14.

We need to study a different Chow variety which takes into account the boundary divisor. Given positive integers $q, m, n, n_1, \dots, n_q, d, d_1, \dots, d_q$ we define the Chow variety $\text{Chow}_m(n, d; n_1, \dots, n_q; d_1, \dots, d_q)$ to be the closed subvariety of

$$\text{Chow}_m(n, d) \times \text{Chow}_m(n_1, d_1) \times \dots \times \text{Chow}_m(n_q, d_q),$$

parametrizing $(q+1)$ -tuples (X, D_1, \dots, D_q) where X is a smooth subvariety of dimension n and degree d in \mathbb{P}^m and each D_i is a smooth subvariety of dimension n_i and degree d_i contained in X .

In [Hwa05], Hwang generalizes the effective bounds on the number of components in [Kol96] to the above defined Chow varieties. We will use his bound to derive an effective estimate on the number of toroidal compactifications.

The first thing to do is to bound the degree of each component of D when embedded into some fixed projective space.

Lemma 3.16. *Let (X, D) be a toroidal compactification. Let d_i be the degree of D_i under the embedding given by Theorem 3.13. Then for any i*

$$d_i \leq (n+1)^{4n-3}(K_X + D)^n.$$

Proof. Let A be the very ample divisor obtained in the proof of Theorem 3.6. Note that A gives the embedding of Theorem 3.13. An easy computation shows that

$$d_i = D_i \cdot A^{n-1} \leq (-1)^{n-1}(n+1)^{3n-3}D^n \leq (n+1)^{4n-3}(K_X + D)^n,$$

where the last inequality is obtained thanks to Lemma 3.9. \square

Finally, we can prove an effective version of Wang's finiteness theorem.

Corollary 3.17. *Fix two positive integers n and V . Then the number of toroidal compactifications with $\dim(X) = n$ and $(K_X + D)^n \leq V$ is less than*

$$\sum_{q=1}^{q_0} d_0^{q+1} \binom{(2n+2)d_0}{2n+1}^{(2n+2)(q+1) \binom{2n+1+d_0}{2n+1}},$$

where $d_0 = (n+1)^{4n}V$ and $q_0 = (2^n + 1)V$.

Proof. Let d be the degree of X in \mathbb{P}^{2n+1} and let d_i be the degree of D_i for any $1 \leq i \leq q$. By definition, there exists a point in $\text{Chow}_{2n+1}(n, d; n-1, \dots, n-1; d_1, \dots, d_q)$ parametrizing (X, D) . Theorem 4.1 in [Fuj92] implies that (X, D) is infinitesimally rigid under deformations of the pair. Therefore, the number of toroidal compactifications with fixed (d, d_1, \dots, d_q) is bounded by the number of components of the corresponding Chow variety. By Theorem 3.13, we can embed X as a smooth subvariety of \mathbb{P}^{2n+1} with degree d bounded by $(n+1)^{4n}V$. Furthermore, each D_i has degree d_i bounded by $(n+1)^{4n-3}V$, by Lemma 3.16. Finally, for each $q > 0$ there are at most d_0^{n+1} choices of (d, d_1, \dots, d_q) , which combined with Proposition 3.2 in [Hwa05] and Theorem 3.11, give the result. \square

Up to constants depending only on n , the above bound is of the form $V^{V^{2n+2}}$.

4. THE TWO DIMENSIONAL CASE

In this section, we substantially improve the theory developed in Section 3 in the special case of surfaces. These improvements rely on two different facts. First, we can derive a optimal version of Theorem 1.1 for $n = 2$. Second, the birational geometry of adjoint linear systems on surfaces is completely understood thanks to the theorem of Reider. Let us recall the statement of this theorem, for more details see [Rei88].

Theorem 4.1 (Reider). *Let X be a smooth surface and let L be a nef line bundle on X . Then the following hold:*

- (1) *If $L^2 \geq 5$ and $x \in X$ is a base point of $|K_X + L|$ then there exists a curve C through x such that either*

$$C \cdot L = 0 \text{ and } C^2 = -1; \text{ or}$$

$$C \cdot L = 1 \text{ and } C^2 = 0.$$

- (2) *If $L^2 \geq 10$ and $x, y \in X$ are two (possibly infinitely near) points which $|K_X + L|$ does not separate then there exists a curve C through x and y such that either*

$$C \cdot L = 0 \text{ and } C^2 \in \{-1, -2\}; \text{ or}$$

$$C \cdot L = 1 \text{ and } C^2 \in \{0, -1\}; \text{ or}$$

$$C \cdot L = 2 \text{ and } C^2 = 0.$$

Let us proceed by deriving an optimal version of Theorem 1.1 when $n = 2$.

Theorem 4.2. *Let (X, D) be a toroidal compactification with $\dim(X) = 2$. Then $K_X + \alpha D$ is ample for any $\alpha \in (\frac{1}{3}, 1)$.*

Proof. By Lemma 2.5, if (X, D) is a toroidal compactification with $\dim(X) = 2$ then each extremal ray must have length one. By Theorem 2.1 in [Mor82], it follows that all the extremal rays are exceptional curves of the first kind, in other words they are smooth rational curves with self-intersection -1 . Let us denote by $\{C_i\}$ the generators of the extremal rays in X . We then have $D \cdot C_i \geq 3$ for all i . If otherwise there would exist a \mathbb{P}^1 with less than three punctures in $X \setminus D$ which is clearly impossible. We then conclude that $(K_X + D) \cdot C_i \geq 2$ for all i . Applying the argument of Proposition 4.12. in [DiC12], we can conclude. \square

As in Section 3, we can now use Theorem 4.2 to give effective very ampleness results and bounds on the number of cusps. We state here the main results. The proofs are similar to the ones obtained in higher dimensions. Thus, we will not give many details. We start by bounding the number of cusps of a two dimensional toroidal compactification.

Proposition 4.3. *Let (X^2, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq \frac{9}{4}(K_X + D)^2.$$

Proof. Because of Theorem 4.2, we know that $3K_X + D = 3L - 2D$ is nef. Since $q \leq -D^2$, by taking the top self-intersection of $3L - 2D$ we get the result. \square

Remark 4.4. *The bound obtained by Hwang in [Hwa04], which seems to be the best published result on this problem, is of the form $q \leq 7.38(K_X + D)^2$. Note that using Proposition 3.12 with $n = 2$, we can also obtain the bound $q \leq 5(K_X + D)^2$.*

Proposition 4.3 gives also a bound of the self-intersection of D in terms of $(K_X + D)^2$. Let us summarize this fact into a corollary.

Corollary 4.5. *Let (X^2, D) be a toroidal compactification. Then*

$$-D^2 \leq \frac{9}{4}(K_X + D)^2.$$

We can now use the theorem of Reider to improve Theorems 1.2 and 1.3 when $n = 2$.

Proposition 4.6. *Let (X^2, D) be a toroidal compactification. Then for all $m \geq 3$ the map associated to $|m(K_X + D)|$ is a morphism which maps all the different components of D to distinct singular points. Furthermore, for all $m \geq 4$ this morphism defines an embedding of $X \setminus D$ into some projective space.*

Proof. Let $L := K_X + D$. Because of Lemma 3.2, we know that $L^2 \geq 3$. By Reider's theorem, $K_X + 2L$ is base point free away from D . Adding D we get that $3(K_X + D)$ is base point free away from D . On the other hand, Lemma 3.1 implies that $2(K_X + D)$ is base point free along D . Similarly, the second statement follows from Reider's theorem applied to $K_X + 3L$. \square

The next proposition exhibits X as a smooth subvariety of \mathbb{P}^5 . Moreover, it gives a bound on its degree.

Proposition 4.7. *Let (X^2, D) be a toroidal compactification. Let $L := K_X + D$. Then $7L - 4D$ is very ample. In particular, $7L - 4D$ embeds X as a smooth subvariety of \mathbb{P}^5 of degree $d \leq 49L^2$.*

Proof. Because of Theorem 4.2, we know that $2K_X + D = 2L - D$ is ample. Theorem 4.1 implies us that $K_X + 3(2L - D) = 7L - 4D$ is very ample. In particular, $7L - 4D$ embeds X as a smooth subvariety of some projective space of degree $d = (7L - 4D)^2 \leq 49L^2$. By general projection we reduce to \mathbb{P}^5 . \square

It remains to bound the degree of each component of D under the embedding described in Proposition 4.7.

Lemma 4.8. *Let d_i be the degree of D_i under the embedding given by $7L - 4D$. Then*

$$d_i \leq 9L^2.$$

Proof. Recall that $D_i \cdot D_j = 0$ if $i \neq j$. Then

$$\begin{aligned} d_i &= D_i \cdot (7L - 4D) = -4D_i \cdot D \\ &= -4D_i^2 \leq -4D^2 \leq 9L^2, \end{aligned}$$

where the last inequality is given by Corollary 4.5. \square

Combining Proposition 4.7 and Lemma 4.8, we can derive a sharper version of Corollary 3.17 for surfaces.

Corollary 4.9. *Fix a positive integer V . Then the number of toroidal compactifications with $\dim(X) = 2$ and $(K_X + D)^2 \leq V$ is bounded by*

$$\sum_{q=1}^{q_0} d_0 (9V)^q \binom{6d_0}{5}^{6(q+1)\binom{5+d_0}{5}},$$

where $d_0 = 49V$ and $q_0 = \lfloor 2.25V \rfloor$.

Proof. Argue as in the proof of Corollary 3.17. \square

Remark 4.10. *The results of this sections improve the ones in [Hwa05]. For example, under the same notation of Proposition 4.7 and Lemma 4.8, he obtains $d \leq 265592 (L^2)^3 + 16059 (L^2)^2 + 242L^2$ and $d_i \leq 394786.1 (L^2)^3$. Finally, the bound in his version of Proposition 4.9 is of the form $V^{V^{16}}$.*

REFERENCES

- [AS95] U. Angehrn and Y.-T. Siu, Effective freeness and point separation for adjoint bundles, *Invent. Math.* **122** (1995), no. 2, 291-308.
- [AMRT10] A. Ash, D. Mumford, M. Rapoport, Y.-S. Tai, Smooth compactifications of locally symmetric varieties. Second edition. Cambridge Mathematical Library. *Cambridge University Press, Cambridge*, 2010.
- [BB66] W. L. Baily, A. Borel, Compactifications of arithmetic quotients of bounded symmetric domains, *Ann. of Math.*, **84** (1966), no.2, 442-528.
- [BJ06] A. Borel, L. Ji, Compactifications of locally symmetric spaces. Mathematics: Theory & Applications, *Birkhäuser Boston, Inc. Boston, MA*, 2006.
- [DM93] P. Deligne, G. Mostow, Commensurabilities among lattices in $PU(1,n)$, *Annals of Mathematics Studies* 132, *Princeton University Press, Princeton, NJ*, 1993.
- [Dem92] J.-P. Demailly, Regularization of closed positive currents and intersection theory, *J. Alg. Geom.* **1** (1992), 361-409.
- [DiC12] G. Di Cerbo, L. F. Di Cerbo, Positivity questions in Kähler-Einstein theory, *arXiv:1210.0218 [mathDG]*, 2012.
- [Ebe96] P. Eberlein, Geometry of nonpositively curved manifolds, *Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL*, 1996.
- [Fuj92] A. Fujiki, An L^2 Dolbeault lemma and its applications, *Publ. Res. Inst. Math. Sci.* **28** (1992), 845-884.
- [Gro82] M. Gromov, *Volume and bounded cohomology*, *Publ. Math. Inst. Hautes Études Sci.* **56** (1982), 5-99.
- [Har77] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York 1977, Graduate Texts In Mathematics, No. 52.
- [Hei02] G. Heier, Effective freeness of adjoint line bundles, *Doc. Math.* **7** (2002), 31-42.
- [Hir71] F. Hirzebruch, The Hilbert modular group, resolution of the singularities at the cusps and related problems. *Séminaire Bourbaki, 23^{me} année (1970/1971), Exp. No. 396*, pp. 275-288. *Lecture Notes in Math.*, Vol. 244, Springer, Berlin, 1971.
- [Hum98] C. Hummel, Rank one lattices whose parabolic isometries have no rotation part. *Proc. Am. Math. Soc.* **126** (1998), 2453-2458.
- [HS96] C. Hummel, V. Schroeder, Cusp closing in rank one symmetric spaces. *Invent. Math.* **123** (1996), 283-307.
- [Hwa04] J.-M. Hwang, *On the volumes of complex hyperbolic manifolds with cusps*, *Internat. J. Math.* **15** (2004), no. 6, 567-572.

- [Hwa05] J.-M. Hwang, *On the number of complex hyperbolic manifolds of bounded volume*, Internat. J. Math. **8** (2005), 863-873.
- [JS93] S. Ji, B. Shiffman, Properties of compact complex manifolds carrying closed positive currents. *J. Geom. Anal.* **3** (1993), 37-61.
- [Kaw97] Y. Kawamata, On Fujita's freeness conjecture for 3-folds and 4-folds, *Math. Ann.* **308** (1997), no. 3, 491-505.
- [Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32. Springer-Verlag, Berlin, 1996.
- [Kol97] J. Kollár, *Singularities of pairs*, Algebraic Geometry, Santa Cruz 1995, Proc. Symp. Pure Math, vol. 62, Amer. Math. Soc., Providence, RI, 1997, 221-287.
- [KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.
- [Laz04a] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48, Berlin: Springer 2004.
- [Laz04b] R. Lazarsfeld, *Positivity in algebraic geometry II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Berlin: Springer 2004.
- [Mar84] G. Margulis, Arithmeticity of the irreducible lattices in semisimple groups of rank greater than 1. *Invent. Math.* **76** (1984), 93-120.
- [Mok12] N. Mok, Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite-volume. Perspective in analysis, geometry, and topology, 331-354, *Prog. Math.*, **296**, Birkhäuser/Springer, New York, 2012.
- [Mor82] S. Mori, Threefolds whose canonical bundle is not numerically effective, *Ann. of Math.* **116** (1982), 133-176.
- [Mum77] D. Mumford, Hirzebruch's Proportionality Theorem in the Non-Compact Case. *Invent. Math.* **42** (1977), 239-272.
- [Par98] J. R. Parker, On the volume of cusped, complex hyperbolic manifolds and orbifolds, *Duke Math. J.* **94** (1998), 433-464.
- [Rei88] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, *Ann. Math.* **127** (1988), 309-316.
- [Sib85] N. Sibony, Quelques problemes de prolongement de courants en analyse complexe. *Duke Math. J.* **52** (1985), 157-197.
- [Tak93] S. Takayama, Ample vector bundles on open algebraic varieties. *Publ. RIMS Kyoto Univ.* **29** (1993), 885-910.
- [Wis89] J. Wiśniewski, Length of extremal rays and generalized adjunction, *Math. Z.* **200** (1989), no. 3, 409-427.
- [SY82] Y. T. Siu, S. T. Yau, Compactification of negatively curved complete Kähler manifolds of finite volume, *Seminars in Differential Geometry*, pp. 363-380, Ann. of Math. Stud., Vol. 102, Princeton Univ. Press, Princeton, N. J., 1982.
- [Wan72] H. C. Wang, *Topics on totally discontinuous groups*, Symmetric Spaces, 459-487, edited by W. B. Boothby and G. L. Weiss, Pure and Appl. Math. 8, Marcel Dekker, New York, 1972.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544-1000, USA
E-mail address: `gdi@math.princeton.edu`

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM NC 27708-0320, USA
E-mail address: `luca@math.duke.edu`